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Fractional charge, spin and statistics of solitons in superfluid ^{3}He film

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Abstract. The topological Chern–Simons term with the parameter $\theta = n\pi$ exists in the superfluid ^{3}He-A thin film, where an integer parameter n may be odd or even depending on the thickness of the film. The particle like soliton in ^{3}He-A film (so-called skyrmion) with integer topological charge Q has spin $s = nQ\hbar/2$ and obeys Fermi–Dirac quantum statistics at odd Q and n. The topological term in action leads to specific quantised Hall effect for spin current in ^{3}He-A. In the planar phase of the ^{3}He film the non-singular 4π spin disclination has the fractional fermion charge $\frac{1}{2}n$ which corresponds to the fractional electric charge $\frac{1}{2}en$ for the disclination in the planar state superconductor. The spin disclination in the ^{3}He-A$_1$ film has both the fractional spin $\frac{1}{4}\hbar n$ and fractional charge $\frac{1}{4}n$.

1. Introduction

In two-dimensional systems with the unit-vector field $\boldsymbol{d}(x, y, t)$ as the order parameter, the hydrodynamical action for $\boldsymbol{d}$ can contain the so called θ-term, or Chern–Simons term, $S_\theta = \hbar\theta H^{\mathrm{Hopf}}$, where H^{Hopf} is the integer-valued Hopf index for $\boldsymbol{d}$-field in (x, y, t) spacetime (Wilczek and Zee 1983). To preserve the unitarity of the $\exp(\mathrm{i}\theta H^{\mathrm{Hopf}})$ it was suggested in Dzyaloshinskii *et al* (1988) that the parameter θ must be $n\pi$ with integer n. This term is responsible both for spin and quantum statistics of topological objects of the field $\boldsymbol{d}$ in (x, y) space, solitons, characterised by an integer-valued Pontryagin index

$$Q = \frac{1}{4\pi}\int \mathrm{d}x\,\mathrm{d}y\,(\boldsymbol{d}\cdot\partial_x\boldsymbol{d}\times\partial_y\boldsymbol{d}). \tag{1.1}$$

A permutation of two identical solitons with topological charges Q changes their linking number and therefore the value of the Hopf invariant by Q^2. As a result the permutation changes the exponent $\exp(\mathrm{i}\theta H^{\mathrm{Hopf}})$ by the factor $(-1)^{nQ^2}$. This means that at odd n the solitons with odd Q behave as fermions under permutation. Also under 2π spin rotation of the $\boldsymbol{d}$ field of the soliton the exponent is multiplied by factor $(-1)^{nQ}$ which means that the spin of the soliton is $s = \hbar nQ/2$ in correspondence with the conventional relation between spin and statistics (Wilczek and Zee 1983).

Such a term was recently proposed for two-dimensional uniaxial Heisenberg antiferromagnets (Dzyaloshinskii *et al* 1988), where it was supposed that for Néel state of spins S on a lattice $n = 2S$. This means that in the case of spin-$\frac{1}{2}$ on a lattice site the soliton with odd Q is a fermion. However the detailed calculations (see e.g. Haldane 1988) showed that for conventional antiferromagnets this term is absent. This also directly follows from the symmetry considerations: the θ-term violates the symmetry of the antiferromagnets, since it changes sign under time inversion.

It was proposed (Volovik 1988a) that the θ-term should exist in the superfluid ^{3}He-A film due to combination of the antiferromagnetic properties of the spin subsystem with ferromagnetic properties of the orbital subsystem of this quasi-two-dimensional condensed matter, since the corresponding breaking of the time and space inversion symmetry allows for the existence of such a term in the hydrodynamical action. However, the method used in Volovik (1988a) for calculation of the θ-value proved to be incorrect.

Here we show that $\theta = n\pi$ for the ^{3}He-A film, where n is the number of the energy levels of transverse quantisation for the Fermi quasi-particles below the Fermi energy, i.e. the maximal n at which the nth transverse energy level $\pi^2 n^2/2a^2$, where a is the film thickness, is still less than the chemical potential μ. This n is roughly proportional to the film thickness a and plays the part of the number of families of fermions in the analogy with the particle physics. Thus the soliton with odd Q is a fermion at odd n and a boson at even n.

In the ^{3}He-A film the unit vector $\boldsymbol{l}$ of the orbital ferromagnetism axis is fixed along the normal $\hat{z}$ to the plane (x, y) of the film, $\boldsymbol{l} = \pm\hat{z}$, while the unit vector $\boldsymbol{d}$ of the spin antiferromagnetism axis is free to rotate, since the spin–orbital interaction in the ^{3}He-A between $\boldsymbol{d}$ and $\boldsymbol{l}$ is negligibly small. Due to the symmetry of the vacuum state of this film the hydrodynamical action for the ^{3}He-A may contain the following term, the θ-term in action, which is expressed in terms of $\boldsymbol{d}$ field through the auxilary 'gauge' field A_μ $(\mu = 0, 1, 2)$:

$$S_\theta = \hbar\theta(\boldsymbol{l}\cdot\hat{z})H^{\mathrm{Hopf}} \qquad H^{\mathrm{Hopf}} = \frac{1}{32\pi^2}\int \mathrm{d}^2x\,\mathrm{d}t\, e^{\mu\nu\lambda}A_\mu F_{\nu\lambda} \tag{1.2}$$

$$F_{\nu\lambda} = \partial_\nu A_\lambda - \partial_\lambda A_\nu = \boldsymbol{d}\cdot\partial_\nu\boldsymbol{d}\times\partial_\lambda\boldsymbol{d}. \tag{1.3}$$

The integrand in (1.2) is odd under time inversion symmetry $(t \to -t)$ and under the orbital rotation by angle π around the x axis ($x \to x$, $y \to -y$, $z \to -z$, this corresponds to the two-dimensional space parity; note also that the $\boldsymbol{d}$-vector which is the vector in spin space is unaffected by this rotation), therefore the θ-term does not exist in conventional antiferromagnets. In the ^{3}He-A film the change of the sign of the integrand under time and space inversion transformations is compensated by the change of the $\boldsymbol{l}$-vector direction into the opposite $(\boldsymbol{l} \to -\boldsymbol{l})$, with the total S_θ being invariant under these symmetry operations.

The integrand in (1.2) is not invariant under the 'gauge' transformation

$$A_\mu \to A_\mu + \partial_\mu\alpha \tag{1.4}$$

while the total integral, the Hopf invariant describing the mapping of the three-dimensional spacetime (x, y, t) onto the sphere S^2 of the unit vector $\boldsymbol{d}$, is invariant under this transformation. The gauge corresponds to the spin rotations by arbitrary angle $\alpha(x, y, t)$ about axis $\boldsymbol{d}$ which do not change the physical state of the system and therefore should not change the hydrodynamical action. Since the gauge is related to the spin rotations the θ-term gives rise to the anomalous terms in the expressions for spin density and spin current density discussed in §3 and in the Appendix.

The gauge property of the integrand results in the following important consequence. The parameter θ can not depend on the space and time coordinates, otherwise the whole integral depends on gauge. Therefore this parameter is fundamental for the ^{3}He-A film and should be constant in certain regions of external parameters, such as film thickness a, and abruptly change to another fundamental value at some critical a. This reminds us of the behaviour of the Hall conductivity σ_{xy} in the quantised Hall effect (QHE). Here it is shown (§2) that the θ-value is defined by the integer topological invariant N which characterises the momentum space topology of the Green function of the ^{3}He-A film of finite thickness. The conservation of this invariant provides the constant value of $\theta = \pi N/2$ at definite regions of the film thickness. In the simplest BCS model of the ^{3}He-A film the topological invariant takes values $N = 2n$ where n is the number of the energy levels of the transverse motion (along the normal to the film) below the Fermi level.

It is also shown (§§3 and 4) that the quantisation of the θ-parameter in the ^{3}He-A film is directly coupled with the quantisation of the Hall conductivity in the analogue of QHE in the ^{3}He-A film discussed in Volovik (1988b). Both have a discontinuity at the same critical values of the film thickness when the diabolical point of the energy spectrum of the Fermi quasi-particles intersects the Fermi level.

In §5 we discuss the other possible superfluid phases in the ^{3}He film: planar state and ^{3}He-A$_1$ phase. In the planar phase of the ^{3}He film, which combines the properties of spin and orbital antiferromagnets, the non-singular 4π spin disclination has the fractional fermion charge $\frac{1}{2}n$ which corresponds to the fractional electric charge $\frac{1}{2}en$ for the disclination in the planar state superconductor. The spin disclinations in the ^{3}He-A$_1$ film, which combines the properties of spin and orbital ferromagnets, have both the fractional spin $\frac{1}{4}\hbar n$ and fractional charge $\frac{1}{4}n$.

2. Calculation of the topological term in action

Here we employ the standard procedure used in quantum field theory (see, for example, Wen *et al* 1988) for the calculation of the topological mass term for the bosonic gauge field $\boldsymbol{\Omega}_\mu$ (which describes the gradients of the ^{3}He-A order parameter $\boldsymbol{d}$) interacting with the system of chiral fermions (Bogoliubov quasi-particles). For the quantum field theory description of superfluid ^{3}He we follow the lectures by Kleinert (1978).

The Bogoliubov fermions in ^{3}He-A film contain four components:

$$\chi = (\psi_\alpha, g_{\alpha\beta}\psi^{\dagger\beta}) \qquad g = \mathrm{i}\sigma^2 \tag{2.1}$$

where $\alpha = \uparrow, \downarrow$ denotes the projection of the ^{3}He nuclear spin, and the antisymmetric metric spinor $g_{\alpha\beta}$ is introduced to provide the identical spinor properties for particle and hole components of χ. The 4×4 matrix Hamiltonian for the Bogoliubov fermions is expressed in terms of the Pauli matrices $\boldsymbol{\sigma}^i$ for ^{3}He nuclear spin and the Pauli matrices $\boldsymbol{\tau}_\mu$ for the Bogoliubov isospin in the particle–hole $\psi, \psi^\dagger$ space:

$$H = \epsilon(\boldsymbol{p})\tau_3 + \tfrac{1}{2}c[(\boldsymbol{p}\cdot\boldsymbol{\tau})(\boldsymbol{d}(x,y,t)\cdot\boldsymbol{\sigma}) + (\boldsymbol{d}(x,y,t)\cdot\boldsymbol{\sigma})(\boldsymbol{p}\cdot\boldsymbol{\tau})]. \tag{2.2}$$

Here $\boldsymbol{p} = (-\mathrm{i}\partial_x, -\mathrm{i}\partial_y)$, $\epsilon(\boldsymbol{p}) = (p^2 - p_{\mathrm{F}}^2)/2m$ and $c = \Delta/p_{\mathrm{F}}$, where p_{F} is the Fermi momentum and Δ is the gap in the quasi-particle energy spectrum on the Fermi circle of the ^{3}He-A film, i.e. at $k_x^2+k_y^2=k_F^2$. The quasi-particle energy spectrum which follows from (2.2) is

$$E^2(k_x, k_y) = H^2 = \epsilon^2(k) + c^2 k^2. \tag{2.3}$$

The effective hydrodynamic action $S\{\boldsymbol{d}(x,y,t)\}$ for the bosonic field $\boldsymbol{d}$ is obtained after the integration over the Fermi field χ which gives:

$$S\{\boldsymbol{d}(x,y,t)\} = \tfrac{1}{2}\mathbf{Tr}\ \ln G \qquad G^{-1} = \mathrm{i}\partial_0 - H$$

$$\mathbf{Tr} = \sum_{\boldsymbol{k},\omega} \mathbf{tr} = \frac{1}{(2\pi)^3}\int \mathrm{d}^2k\,\mathrm{d}\omega\,\mathrm{d}^2x\,\mathrm{d}t\,\mathbf{tr} \tag{2.4}$$

where $\mathbf{tr}$ means the trace over both spin and Bogoliubov isospin indices; the factor $\frac{1}{2}$ compensates for the double summation caused by introduction of particles and holes in the system of the ^{3}He particles.

For the calculation of the topological term in action it is more convenient to use instead of the unit vector $\boldsymbol{d}$ field the spin rotation 2×2 matrix $\mathbf{U}(x,y,t)$ which corresponds to the rotation of the $\boldsymbol{d}$ field from some homogeneous field $\boldsymbol{d}=\hat{\boldsymbol{z}}$

$$\boldsymbol{d}(x,y,t)\cdot\boldsymbol{\sigma} = \hat{\boldsymbol{z}}\cdot\mathbf{U}^{-1}(x,y,t)\boldsymbol{\sigma}\mathbf{U}(x,y,t). \tag{2.5}$$

Then after the unitary transformation of the Fermi field, $\chi\to\mathbf{U}^\dagger\chi$, the effective hydrodynamic action is expressed in terms of the gauge field

$$\mathbf{A}_\mu = \mathrm{i}\mathbf{U}\partial_\mu\mathbf{U}^\dagger = \tfrac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\Omega}_\mu \qquad \mu=(0,1,2) \tag{2.6}$$

$$\begin{aligned} &S\{\mathbf{A}(x,y,t)\} = \tfrac{1}{2}\mathbf{Tr}\ \ln\tilde{G}(p_\mu - \mathbf{A}_\mu) \\ &\tilde{G}^{-1}(k_\mu) = k_0 - \epsilon(\boldsymbol{k})\tau_3 - c\sigma^3(k_x\tau_1 + k_y\tau_2). \end{aligned} \tag{2.7}$$

In what follows we shall omit the tilde accent on $\tilde{G}$.

In terms of this gauge field the Hopf invariant (1.2) is

$$H^{\mathrm{Hopf}} = \frac{1}{96\pi^2}\int \mathrm{d}^2x\,\mathrm{d}t\,e^{\mu\nu\lambda}\boldsymbol{\Omega}_\mu\cdot\boldsymbol{\Omega}_\nu\times\boldsymbol{\Omega}_\lambda. \tag{2.8}$$

This may be seen directly by inserting $\boldsymbol{\Omega}_\mu$ expressed in terms of the gradients of the $\boldsymbol{d}$ field and A_μ, which corresponds to the rotation about axis $\boldsymbol{d}$:

$$\boldsymbol{\Omega}_\mu = -\boldsymbol{d}A_\mu + \boldsymbol{d}\times\partial_\mu\boldsymbol{d}. \tag{2.9}$$

The topological term in (2.8) is obtained from the $\mathbf{AAA}$ and $\mathbf{A}\partial\mathbf{A}$ terms of the expansion of $S\{\mathbf{A}(x,y,t)\}$. The triangle diagram gives for the cubic term $\mathbf{AAA}$ the following expression:

$$S_3 = \frac{1}{6}\mathbf{Tr}\left(G\frac{1}{2}\left\{\frac{\partial G^{-1}}{\partial k_\mu},\mathbf{A}_\mu\right\}_+\right)^3 = b_{\mu\nu\lambda}\int \mathrm{d}^2x\,\mathrm{d}t\,\Omega^1_\mu\Omega^2_\nu\Omega^3_\lambda \tag{2.10}$$

where the calculation of the anticommutators $\{,\}_+$ using the σ^3-dependence of G in (2.7) results in

$$b_{\mu\nu\lambda} = \frac{i}{8}\mathbf{tr}\sum_{\boldsymbol{k}\omega} G(\Delta)\frac{\partial G^{-1}(0)}{\partial k_\mu}G(-\Delta)\frac{\partial G^{-1}(0)}{\partial k_\nu}G(\Delta)\frac{\partial G^{-1}(\Delta)}{\partial k_\lambda}. \tag{2.11}$$

Here $G(\Delta)$ means the dependence of G on the gap Δ with $G(0) \equiv G(\Delta = 0)$.

The two-vertex loop diagram for the $\mathbf{A}\partial\mathbf{A}$ term gives

$$\begin{aligned} S_2 &= \frac{-i}{4}\mathbf{Tr}\left(G\frac{1}{2}\left\{\frac{\partial G^{-1}}{\partial k_\mu}, \mathbf{A}_\mu\right\}_+ G\frac{\partial G^{-1}}{\partial k_\nu}G\frac{1}{2}\left\{\frac{\partial G^{-1}}{\partial k_\lambda}, \partial_\nu \mathbf{A}_\lambda\right\}_+\right) \\ &= -\frac{1}{2}\int d^2x\, dt\, [b_{\mu\nu\lambda}(\Omega^1_\nu\partial_\lambda\Omega^1_\mu + \Omega^2_\nu\partial_\lambda\Omega^2_\mu) + c_{\mu\nu\lambda}\Omega^3_\nu\partial_\lambda\Omega^3_\mu] \end{aligned} \tag{2.12}$$

where

$$c_{\mu\nu\lambda} = \frac{i}{8}\mathbf{tr}\sum_{\boldsymbol{k}\omega} G\partial_{k_\mu}G^{-1}G\partial_{k_\nu}G^{-1}G\partial_{k_\lambda}G^{-1}. \tag{2.13}$$

The summation of two diagrams leads to the cancellation of the terms with $b_{\mu\nu\lambda}$ due to relation $b_{\mu\nu\lambda} = -b_{\nu\mu\lambda}$, which follows from the symmetry of the ³He-A under spin rotation about axis z, and due to the fact that the field $\boldsymbol{\Omega}_\mu$ is pure gauge field, i.e.

$$\boldsymbol{f}_{\mu\nu} = \partial_\mu\boldsymbol{\Omega}_\nu - \partial_\nu\boldsymbol{\Omega}_\mu - \boldsymbol{\Omega}_\mu \times \boldsymbol{\Omega}_\nu = 0. \tag{2.14}$$

As a result these two diagrams produce the Hopf term H^{Hopf} (see (2.8)) in the hydrodynamical action

$$S = \theta H^{\mathrm{Hopf}} \qquad \theta = \pi N/2 \tag{2.15}$$

where N is the momentum space topological invariant

$$N = \frac{1}{24\pi^2}e^{\mu\nu\lambda}\int dk_x\, dk_y\, d\omega\, \mathbf{tr}\, G\partial_{k_\mu}G^{-1}G\partial_{k_\nu}G^{-1}G\partial_{k_\lambda}G^{-1}. \tag{2.16}$$

This integer invariant describes the non-trivial mapping of the three-dimensional $(\boldsymbol{k}, \omega)$ space into the space of the non-degenerate 4×4 matrices G and therefore the result (2.15) for θ does not depend on the details of the system. This justifies the application of the BCS model for ³He-A: both ³He-A and its BCS model have the same topological structure for the Green function and therefore should give the same θ-term. According to Volovik (1988b) for the pure two-dimensional ³He-A this invariant is $N = 2$, since the spin-up and spin-down components of ³He-A give equal contributions to N, while for the BCS model of the ³He-A film of finite thickness each level of the transverse motion gives the same contribution to N, therefore one has $N = 2n$ where n is the number of the energy levels in the film below the Fermi energy, which plays the part of the number of the families of fermions in particle physics. In the real ³He-A film the interaction between different transverse energy levels may be important and the even-integer N is given by general expression

$$N = \frac{1}{24\pi^2}\int \mathbf{Tr}\, \mathbf{G}\partial\mathbf{G}^{-1} \wedge \mathbf{G}\partial\mathbf{G}^{-1} \wedge \mathbf{G}\partial\mathbf{G}^{-1} \tag{2.17}$$

where the Green function G_{mn} also contains transverse-level indices

$$G_{mn}^{-1}(\boldsymbol{k}, \omega) = \mathrm{i}\omega\delta_{mn} - \epsilon_{mn}(\boldsymbol{k})\tau_3 - c_{mn}(\boldsymbol{k}\cdot\boldsymbol{\tau})(\boldsymbol{d}(x, y, t)\cdot\boldsymbol{\sigma}). \tag{2.18}$$

Only in the limit of the small interaction between the transverse energy levels does the invariant N coincide with the double number of the levels below the Fermi energy. However, in any case $\theta = \pi n$ where n is odd or even depending on the film thickness, and abruptly changes at some critical values of the film thickness when the diabolical points in the Fermi quasi-particle spectrum intersect the Fermi level (see Volovik 1988b). Note that in A_1 phase the topological invariant N may be odd leading to the parastatistics for the topological objects (see §5).

Note also that there are no technical difficulties in calculating the diagrams. As distinct from the relativistic quantum field theory all the diagrams are convergent at large momenta and no cut-off problem arises. On the other hand there are no infrared singularities which are crucial for the properties of the bulk ^{3}He-A due to zeros in the quasi-particle energy gap at two points on the Fermi sphere: the quasi-particle energy in the ^{3}He-A film, (2.3), has no zeros due to quantisation of transverse motion.

3. Analogue of quantum Hall effect for spin current

To calculate spin density it is usually convenient to introduce an external magnetic field $\boldsymbol{H}(x, y, t)$ and find the response on this field. This field produces the term $\frac{1}{2}\gamma\boldsymbol{\sigma}\boldsymbol{H}$ in the Green function, which corresponds to the $\mathbf{A}_0$ component of the external SU(2) gauge field. In the same manner, to calculate the spin current density that follows from the existence of the anomalous θ-term in the hydrodynamical action it is convenient to introduce the components $\mathbf{A}_1$ and $\mathbf{A}_2$ of the gauge field.

Therefore it is instructive to introduce the general external SU(2) gauge field $\boldsymbol{\Omega}_\mu^{\mathrm{ext}}$ with non-zero curvature $\boldsymbol{f}_{\mu\nu}^{\mathrm{ext}}$ together with the spacetime dependence of the $\boldsymbol{d}$ field . It can be done in such a manner that the Green function G shall acquire the invariance under local SU(2) spin rotations of all the fields, $\boldsymbol{\Omega}_\mu^{\mathrm{ext}}$, $\boldsymbol{d}$ and χ .

The effective hydrodynamical action for $\boldsymbol{\Omega}_\mu^{\mathrm{ext}}$ and $\boldsymbol{d}$ may be again obtained by the spin rotation $\mathbf{U}$ of the $\boldsymbol{d}$ to a constant field $\hat{\boldsymbol{z}}$. After this rotation we come to the same action and Green function for fermions as in (2.7), where the pure gauge field $\mathbf{A} = \mathrm{i}\mathbf{U}\partial\mathbf{U}^{-1}$ should be substituted by $\mathbf{A}^{\mathrm{tot}} = \mathbf{U}\mathbf{A}^{\mathrm{ext}}\mathbf{U}^{-1} + \mathrm{i}\mathbf{U}\partial\mathbf{U}^{-1}$.

Now we may expand the effective action in series of $\boldsymbol{\Omega}_\mu^{\mathrm{tot}}$ in the same manner as previously but taking into account that $\boldsymbol{f}_{\mu\nu}^{\mathrm{tot}}$ is non-zero. Then expressing $\mathbf{U}\partial\mathbf{U}^{-1}$ in terms of the gradients of the $\boldsymbol{d}$ one obtains the following anomalous part of the action for the two-dimensional ^{3}He-A film in terms of $\boldsymbol{\Omega}_\mu^{\mathrm{ext}}$ and $\boldsymbol{d}$ which is invariant under the local SU(2) gauge transformation:

$$S = S_1(\boldsymbol{\Omega}_\mu^{\mathrm{ext}}) + S_2(\boldsymbol{d}) + S_{12} + S_{\mathrm{surf}} \tag{3.1}$$

where $S_1(\boldsymbol{\Omega}_\mu^{\mathrm{ext}})$ is the Chern–Simons term for the external gauge field:

$$S_1(\boldsymbol{\Omega}_\mu^{\mathrm{ext}}) = \frac{1}{32\pi}\int \mathrm{d}^2x\,\mathrm{d}t\, e^{\mu\nu\lambda}(\tfrac{1}{3}\boldsymbol{\Omega}_\mu^{\mathrm{ext}}\cdot\boldsymbol{\Omega}_\nu^{\mathrm{ext}}\times\boldsymbol{\Omega}_\lambda^{\mathrm{ext}} + \boldsymbol{\Omega}_\mu^{\mathrm{ext}}\cdot\boldsymbol{f}_{\nu\lambda}^{\mathrm{ext}}). \tag{3.2}$$

The $S_2(\boldsymbol{d})$ is the Chern–Simons term for the $\boldsymbol{d}$ field

$$S_2(\boldsymbol{d}) = \frac{1}{32\pi}\int \mathrm{d}^2x\,\mathrm{d}t\, e^{\mu\nu\lambda}A_\mu F_{\nu\lambda} \tag{3.3}$$

and the interaction of the external field with the order parameter in the anomalous part of action is as follows:

$$S_{12} = \frac{1}{32\pi} \int \mathrm{d}^2x\, \mathrm{d}t\, [e^{\mu\nu\lambda}(\partial_\mu \boldsymbol{d} - \boldsymbol{\Omega}^{\mathrm{ext}}_\mu \times \boldsymbol{d}) \cdot (\boldsymbol{f}^{\mathrm{ext}}_{\nu\lambda} \times \boldsymbol{d}) + 2(\partial_0 \boldsymbol{d} - \boldsymbol{\Omega}^{\mathrm{ext}}_0 \times \boldsymbol{d}) \cdot (\boldsymbol{f}^{\mathrm{ext}}_{12} \times \boldsymbol{d})]. \tag{3.4}$$

The surface term may be also useful, e.g. for the calculation of the total spin of the soliton (see Appendix) since far from the soliton the field A_i decreases slowly. Here we leave only the part which is responsible for the spin of soliton:

$$S_{\mathrm{surf}} = \frac{1}{16\pi} \int \mathrm{d}S_i\, e_{ij0} \boldsymbol{d} \cdot \boldsymbol{\Omega}^{\mathrm{ext}}_0 A_j. \tag{3.5}$$

The above equations are useful for the calculation of different quantities. Here we are interested in the spin current response on the gradient of magnetic field, which is reminiscent of the response of the particle current on the electric field in QHE. We consider the homogeneous $\boldsymbol{d} = \hat{z}$ field and consider the spin current j_i^x, i.e. the current of the x-component of spin in the presence of magnetic field. To calculate this current one must leave only the terms that are linear both in Ω_i^1 and in $\boldsymbol{\Omega}_0 = \boldsymbol{H}$, then the variation of (3.1) over Ω_i^1 gives

$$j_i^x = \tfrac{1}{16}\pi l_k e_{ijk} \partial_j (\gamma H^x). \tag{3.6}$$

For the general case of the film with finite thickness the factor in the response of spin current on the gradient of magnetic field is quantised in terms of the topological invariant N in (2.17):

$$j_i^x = (N/32\pi) l_k e_{ijk} \partial_j (\gamma H^x) \tag{3.7}$$

where $N = 2n$ in the limit of small interaction of the n transverse energy levels below the Fermi energy. This spin current QHE may be also obtained in another way due to its relation with QHE in ^{3}He-A for mass current.

4. Spin current from the QHE for particle current

The anomalous spin current may be found from the anomalous particle current if one uses the representation of the ^{3}He-A in terms of two superfluid components with spin up and spin down (see the review by Leggett (1975)). In this representation the spins of the particles are perpendicular to the $\boldsymbol{d}$-vector. The advantage of this picture of the ^{3}He-A is that the spin current is just the counterflow of two components, $\frac{1}{2}\hbar(j_\uparrow - j_\downarrow)$.

According to Volovik (1988b), for each component there exists the analogue of the quantised Hall effect (QHE): in the applied gradient of the chemical potential, $\partial\mu$, which for the electrically neutral liquid plays the part of the electric field, the transverse particle current appears with the quantised Hall conductivity $\sigma_{xy} = n/4h$:

$$j^i_\uparrow = \tfrac{1}{2}\sigma_{xy} e^{ijk} l_{\uparrow j} \partial_k \mu_\uparrow \qquad j^i_\downarrow = \tfrac{1}{2}\sigma_{xy} e^{ijk} l_{\downarrow j} \partial_k \mu_\downarrow \tag{4.1}$$

$$\sigma_{xy} = n/4\pi\hbar. \tag{4.2}$$

Here again the integer $\tilde{n}$ is the number of the energy levels of the particle motion in z direction below the Fermi energy, n depends on the film thickness a and is roughly proportional to a. As distinct from the conventional QHE in a two-dimensional system of electrons this is the anomalous QHE in the sense that it takes place without any magnetic field and is produced by the orbital ferromagnetism of the ^{3}He-A, i.e. instead of the magnetic field direction the direction of the orbital momentum, $\boldsymbol{l}$, defines the direction of the Hall current. In ^{3}He-A the superfluid components have the same orbital momentum: $\boldsymbol{l}_\uparrow = \boldsymbol{l}_\downarrow = \boldsymbol{l}$. The electric charge is not involved in (4.2) since we consider the current of particles instead of the electric current.

In the applied magnetic field, which in neutral ^{3}He liquid interacts with ^{3}He nuclear spins only, the chemical potentials for the components split. If the magnetic field is directed along the spin quantisation axis then

$$\mu_\uparrow = \mu - \tfrac{1}{2}\hbar\gamma H \qquad \mu_\downarrow = \mu + \tfrac{1}{2}\hbar\gamma H. \tag{4.3}$$

As a result the current of the spin projection on the magnetic field arises:

$$\tfrac{1}{2}\hbar(j^i_\uparrow - j^i_\downarrow) = -(n\hbar/16\pi)e^{ijk}l_j\gamma\partial_k H. \tag{4.4}$$

To write down the general vector form of the spin current one must take into account that the spin quantisation axis in two component representation of the ^{3}He-A is perpendicular to the $\boldsymbol{d}$-vector. Therefore in the general expression for the spin current one must change $H \rightarrow \boldsymbol{H}_\perp = \boldsymbol{H} - \boldsymbol{d}(\boldsymbol{d}\cdot\boldsymbol{H})$:

$$\boldsymbol{j}^i_{\rm spin} = -(n\hbar/16\pi)e^{ijk}l_j\gamma\partial_k[\boldsymbol{H} - \boldsymbol{d}(\boldsymbol{H}\cdot\boldsymbol{d})] \tag{4.5}$$

which coincides with (3.7).

The quantisation of the spin current under the gradient of external magnetic field is better to observe in the oscillating regime of NMR. The possibility of using the powerful magnetic methods makes the measurement of the QHE on spin current in the ^{3}He-A film more preferable than that on the mass current.

5. Fractional charge in the planar state

In the ^{3}He film two superfluid phases are possible in equilibrium: ^{3}He-A and a planar state (see ,e.g., Brusov and Popov 1981). Also the A_1 state should exist in a large applied magnetic field or possibly for the film on magnetic substrate. We first consider the anomalous properties of the planar state.

In the two spin component representation (see the previous section) the planar state has two superfluid components with the opposite spin projections, $s_\uparrow = -s_\downarrow = \hat{s}$, and also with the opposite orbital momenta: $\boldsymbol{l}_\uparrow = -\boldsymbol{l}_\downarrow = \boldsymbol{l}$. Therefore the time inversion symmetry is not broken in this state; however, the symmetry of this phase allows for the response of particle current on the gradient of magnetic field. According to the results of the previous section this response should be quantised:

$$j^i = (j^i_\uparrow + j^i_\downarrow) = -(n/8\pi)e^{ijk}\boldsymbol{l}_j\gamma\partial_k(\boldsymbol{H}\cdot\hat{s}). \tag{5.1}$$

This points out that in the planar state there should be also the anomalous term in the hydrodynamical action with the quantised parameter due to the topological invariant in momentum space. One may expect the following term:

$$S_\theta = \frac{\alpha}{16\pi}\int \mathrm{d}^2x\,\mathrm{d}t\, e^{\mu\nu\lambda}B_\mu F_{\nu\lambda} \tag{5.2}$$

$$F_{\nu\lambda} = \partial_\nu A_\lambda - \partial_\lambda A_\nu = \hat{s}\cdot\partial_\nu\hat{s}\times\partial_\lambda\hat{s} \tag{5.3}$$

where A_μ is the gauge field corresponding to the spin rotation about axis $\hat{s}$ and B_μ is the U(1) gauge field corresponding to the phase rotation of the order parameter, or the electromagnetic field in superconductors; α is the parameter to be quantised.

The variation of (5.2) over B_i produces the particle current

$$j_i = (\alpha/8\pi)e^{ijk}\boldsymbol{l}_j(\partial_k A_0 - \partial_0 A_k) \tag{5.4}$$

which fits (5.1) if $\alpha = n$ (A_0 corresponds to $(\boldsymbol{H}\cdot\hat{s})$). With this value of α one may find the density of particles

$$\rho = \delta S/\delta B_0 = (n/8\pi)F_{12}. \tag{5.5}$$

Though the above procedure for derivation of (5.5) is not quite correct the validity of this equation is confirmed by direct calculation of the particle density (5.5) from the expansion of $\rho = \frac{1}{2}\mathbf{Tr}\,\tau_3 G$ in terms of the gradients of the order parameter. On the other hand the direct calculation of the term in the action of the type of equation (5.2) in the weak coupling approximation gives for this term the result different from (5.2):

$$S_\theta = \frac{\alpha}{8\pi}\int \mathrm{d}^2x\,\mathrm{d}t\,e^{0ij}(B_0\partial_i A_j + B_i\partial_j A_0). \tag{5.2a}$$

The only difference from (5.2) is that this term in (5.2a) does not contain the time derivatives. This difference however does not influence the results (5.1) and (5.5) for particle current and particle density, obtained by variation of (5.2a).

The spatial integration of (5.5) leads to the fractional fermionic charge

$$\int \mathrm{d}^2x\ \ \rho = nQ/2 \tag{5.6}$$

for the non-singular disclination with topological charge

$$Q = \frac{1}{4\pi}\int \mathrm{d}^2x\,F_{12} \tag{5.7}$$

It is worthwhile noting that the momentum space topological invariant which is responsible for the quantisation of the parameter in the expressions for particle density and particle current in the planar state is

$$\tilde{N} = \frac{1}{24\pi^2}\int \mathbf{Tr}\,\tau_3\sigma^3\mathbf{G}\partial\mathbf{G}^{-1}\wedge\mathbf{G}\partial\mathbf{G}^{-1}\wedge\mathbf{G}\partial\mathbf{G}^{-1}. \tag{5.8}$$

For the planar phase one has $\tilde{N} = 2n$ if the interaction between n transverse levels below the Fermi level is small enough and $N = 0$ contrary to the ^{3}He-A where $N = 2n$ and $\tilde{N} = 0$. In terms of these two topological invariants in momentum space one may write the spin s and charge q of the soliton in general case of both superfluid phases:

$$s = \hbar NQ/4 \qquad q = e\tilde{N}Q/4. \tag{5.9}$$

The A_1 phase contains only one spin component with the Cooper pair spin $\hat{s}$ and orbital momentum $\boldsymbol{l}$. This phase is the combination of the spin and orbital ferromagnets.

The particle current and spin current coincide in this phase. The momentum space topological invariants for this state are $\tilde{N} = N = n$. As a result, according to (5.9) the non-singular disclination with the topological charge Q in (5.7) has both fractional spin and fractional charge with the magnitudes twice less than in ^{3}He-A and planar phase correspondingly:

$$\int \mathrm{d}^2 x\, \rho = nQ/4 \qquad \int \mathrm{d}^2 x\, \boldsymbol{S} = n\hbar Q\hat{s}/4. \tag{5.10}$$

6. Discussion

It is found that the quantum statistics of the solitons in ^{3}He-A film essentially depends on the film thickness, abruptly changing at some critical values of thickness. At the moment of transition between Fermi and Bose statistics the quasi-particle energy gap disappears, i.e. the system passes through the dissipative state, in a complete analogy with QHE (see Volovik 1988b for details). The topological θ-term in action, which is responsible for quantum statistics, also leads to the quantum Hall effect for spin current.

The other superfluid phases in ^{3}He films also may exhibit the quantisation of parameters, with fractional spin and fermionic charge. All these phases combine the properties of spin and orbital magnets in different ways resulting in different properties of topological objects and different types of QHE. For example the spin disclination in (i) ^{3}He-A, which is spin antiferromagnet and orbital ferromagnet, has fractional spin but no fermionic charge; in (ii) planar phase, which is spin and orbital antiferromagnets, the corresponding spin disclination has the fractional fermionic charge without spin; in (iii) ^{3}He-A$_1$ phase, which is the combination of spin and orbital ferromagnets, this spin disclination has both fractional spin and fractional charge.

In the relation of the fractional charge and spin of the solitons the ^{3}He film should not be unique. Among the magnets there should exist the quasi-two-dimensional electron systems where the spin ferro- or antiferromagnetism is combined with the orbital ferro- or antiferromagnetism in such a way that the symmetry allows for either the θ-term in action and therefore fractional statistics of solitons or the fractional charge for topological objects. The possibility of existence of neutral objects obeying fractional statistics and charge-e bosons was proposed in the resonating-valence-bond state (see, e.g., Zou and Anderson 1988). The analogy between the ground states of the magnets and the fractional quantum Hall states (Laughlin 1983) with respect to fractional statistics was discussed in Kalmeyer and Laughlin (1987) and Arovas *et al* (1988).

Also the three-dimensional magnetic structures can exist which have five- or four-dimensional topological terms in action (see, e.g., Witten 1983, Balachandran 1986). This may give both the Fermi statistics and fractional electric charge for three-dimensional particle-like solitons in magnets and unconventional superconductors.

Appendix. Anomalous spin density in ^{3}He-A film

The anomalous local spin density is obtained as response of the anomalous action, (3.1), on the magnetic field $\boldsymbol{H} = \boldsymbol{\Omega}_0^{\text{ext}}$ in the limit $\boldsymbol{\Omega}_\mu^{\text{ext}} \to 0$. If the surface term is not important the variation of (3.3) gives for local spin density:

$$s(x, y) = \delta S/\delta \mathbf{\Omega}_0^{\rm ext} = (\hbar N/32\pi) l_k e_{ijk} \partial_i \boldsymbol{d} \times \partial_j \boldsymbol{d}. \tag{A1}$$

This spin density is a derivative and therefore does not contribute to the total spin of the soliton. Therefore the spin of the soliton is defined completely by the surface terms or by the asymptotic behaviour of the gauge fields far from the soliton. However, the local spin density has remarkable property: it is directed along the local vector $\boldsymbol{d}$, which serves as a local axis of spin quantisation for excitations in ^{3}He-A, and its value is proportional to the density $(1/4\pi)F_{12}$ of the topological charge of the soliton, (1.1). The integral of the spin projection on the quantisation axis means the total spin for internal observer, the observer who lives in the ^{3}He-A vacuum. The integration of the spin projection over the soliton cross section is thus the spin of the soliton from the point of view of the internal observer

$$\tilde{s} = \int \mathrm{d}^2x \;\; s\boldsymbol{d} = \hbar N Q/4 = (\hbar/2\pi)\theta Q \tag{A2}$$

is thus the spin of the soliton from the point of view of the internal observer. This spin also satisfies the relation between spin and quantum statistics of the soliton.

The same expression (A1) for the local spin density may be obtained from the variation of the θ term, (1.2). Alongside the gauge field A_μ one may introduce the corresponding (2+1)-current J^μ:

$$J^\mu = \delta S_\theta/\delta A_\mu = (\hbar\theta/8\pi^2)(\boldsymbol{l}\hat{\boldsymbol{z}}) e^{\mu\nu\lambda} F_{\nu\lambda} \qquad \partial_\mu J^\mu = 0. \tag{A3}$$

Since the gauge is related to the spin rotations around the local axis $\boldsymbol{d}$ this current is related to the spin density projection on the local axis $\boldsymbol{d}$: $\boldsymbol{s}\cdot\boldsymbol{d} = J^0$.

From the θ-term one may also obtain the total spin of the soliton including the surface contribution. According to the Noether theorem the spin density $\boldsymbol{S}$ and the spin current density $\boldsymbol{j}^i_{\rm spin}$, related by spin conservation law $\partial_t \boldsymbol{S} + \partial_i \boldsymbol{j}^i_{\rm spin} = 0$, $(i = 1, 2)$, are obtained from the Lagrangian by space- and time-dependent spin rotation by solid angle $\boldsymbol{\theta}$ with subsequent differentiation of the Lagrangian by $\mathbf{\Omega}_\mu = \partial_\mu \boldsymbol{\theta}$.

Under spin rotations the gauge field transforms as follows:

$$A_\mu \to A_\mu - (\boldsymbol{d} - \boldsymbol{d}(\infty)) \cdot \partial_\mu \boldsymbol{\theta} \qquad F_{\mu\nu} \to F_{\mu\nu} + \partial_\mu \boldsymbol{\theta} \cdot \partial_\nu \boldsymbol{d} - \partial_\nu \boldsymbol{\theta} \cdot \partial_\mu \boldsymbol{d}. \tag{A4}$$

Here an additional dependence on $\boldsymbol{d}(\infty)$ is introduced to eliminate the constant gauge field at infinity, since (1.2) for the Hopf invariant is valid only for the gauge field vanishing at infinity. This leads to the following spin density $(i = 1, 2, 3)$:

$$\begin{aligned} \boldsymbol{S} &= -(\hbar\theta/32\pi^2) e^{ijk} l_k [(\boldsymbol{d} - \boldsymbol{d}(\infty)) F_{ij} + 2A_i \partial_j \boldsymbol{d}] \\ &= \boldsymbol{s} + (\hbar\theta/32\pi^2) e^{ijk} l_k [\boldsymbol{d}(\infty) F_{ij} + 2\partial_j (A_i \boldsymbol{d})] \end{aligned} \tag{A5}$$

where the first term $\boldsymbol{s}$ is the local spin in (A1) while the other is the contribution of the asymptotes. The integration of the second term over the soliton gives for the spin of the soliton

$$\boldsymbol{S} = \int \mathrm{d}^2x\, \boldsymbol{S}(x, y) = (\hbar\theta Q/2\pi)\boldsymbol{d}(\infty) \tag{A6}$$

in accordance with the relation between spin and statistics.

Note that each part of the second term gives the half of the result in (A6). The first part comes from the gauge, while the second part is from the surface term in (3.5). In principle the total spin of the soliton comes from the surface term and corresponds to the half of the value in equation (A6). But according to Kivelson and Rokhsar (1988) the 'natural' statistics are those that eliminate the gauge forces between the solitons. The first part of the second term of (A5) is introduced simply to cancel the gauge interaction between the solitons. Together both parts of the second term produce the 'natural' statistics corresponding to the spin $\frac{1}{2}$ of the elementary soliton with $Q = \pm 1$.

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